arXiv:math/0409009v1 [math.CV] 1 Sep 2004

A family of Schottky groups arising from the hypergeometric equation (tex/schot/IY2)

Takashi ICHIKAWA * Masaaki YOSHIDA †

February 1, 2008

Abstract: We study a complex 3-dimensional family of classical Schottky groups of genus 2 as monodromy groups of the hypergeometric equation. We find non-trivial loops in the deformation space; these correspond to continuous integer-shifts of the parameters of the equation.
Keywords: hypergeometric equation, monodromy group, Schottky group
MSC: 33C05, 30F10, 30F40
Running title: Schottky groups arising from the hypergeometric equation

Contents

1 Introduction

When the three exponents of the hypergeometric differential equation are purely imaginary, its monodromy group is a classical Schottky group of genus 2; such

*Department of Mathematics, Faculty of Science and Engineering, Saga University, Saga 840-8502 Japan, *E-mail address*: ichikawa@ms.saga-u.ac.jp

†Department of Mathematics, Kyushu University, Fukuoka 810-8560 Japan, *E-mail address*: myoshida@math.kyushu-u.ac.jp

groups form a real 3-dimensional family. Since, under a small deformation, a Schottky group remains to be a Schottky group, by deforming the parameters of the hypergeometric equation, we have a complex 3-dimensional family of Schottky groups. We introduce (in §4) a complex 3-dimensional family S of classical Schottky groups, containing the above ones coming from pure-imaginary-exponts cases, equipped with some additional structure. We study a structure of S, and construct loops (real 1-dimensional families of classical Schottky groups) in S generating the fundamental group of S. These loops correspond to *continous* interger-shifts of the parameters of the hypergeometric equation.

2 The hypergeometric equation

We consider the hypergeometric differential equation

$$E(a,b,c) : x(1-x)\frac{d^2u}{dx^2} + \{c-(a+b+1)x\}\frac{du}{dx} - abu = 0.$$

For (any) two linearly independent solutions u_1 and u_2, the (multi-valued) map

$$s : X := \mathbf{C} - \{0,1\} \ni x \longmapsto z = u_1(x) : u_2(x) \in \mathbf{P}^1 := \mathbf{C} \cup \{\infty\}$$

is called a Schwarz map (or Schwarz's s-map). If we choose as solutions u_1 and u_2, near $x=0$, a holomorphic one and x^{1-c} times a holomorphic one, respectively, then the circuit matrices around $x=0$ and 1 are given by

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(1-c)} \end{pmatrix} \quad \text{and} \quad \gamma_2 = P^{-1}\begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(c-a-b)} \end{pmatrix} P,$$

respectively, where P is a connection matrix given by

$$P = \begin{pmatrix} \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} & \dfrac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} \\ \dfrac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} & \dfrac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} \end{pmatrix};$$

here Γ denotes the Gamma function. Note that the matrices act the row vector (u_1, u_2) from the right. These generate the monodromy group $M(a,b,c)$ of the equation $E(a,b,c)$.

3 Hypergeometric equations with purely imaginary exponent-differences ([SY], [IY])

If the exponent-differences

$$\lambda = 1-c, \quad \mu = c-a-b, \quad \nu = b-a$$

(at the singular points $x = 0, 1$ and ∞, respectively) are purely imaginary, the image of the upper-half part

$$X^+ := \{x \in X \mid \Im x \geq 0\}$$

is bounded by the three circles (in the s-plane) which are images of the three intervals $(-\infty, 0), (0, 1)$ and $(1, +\infty)$; put

$$C_1 = s((-\infty, 0)), \quad C_3 = s((0, 1)), \quad C_2 = s((0, +\infty)).$$

Note that if we continue analytically s through the interval, say $(0, 1)$, to the lower-half part $X^- := \{x \in X \mid \Im x \leq 0\}$, then the image of X^- is the mirror image of that of X^+ under the reflection with center C_3.

The reflection with center (mirror) C_j will be denoted by ρ_j $(j = 1, 2, 3)$. The monodromy group $M(a, b, c)$ is the group consisting of even words of ρ_j $(j = 1, 2, 3)$. This group is a Schottky group of genus 2. The domain of discontinuity modulo $M(a, b, c)$ is a curve of genus 2 defined over reals.

Choosing the solutions u_1 and u_2 suitably, we can assume that the centers of the three circles are on the real axis, and that the circles C_2 and C_3 are inside the circle C_1 (see Figure 1). Let C_1' and C_2' be the mirror images with respect to C_3 of the

Figure 1: The circles and the disks

circles C_1 and C_2, respectively. Then $\gamma_1 := \rho_3\rho_1$ maps inside of C_1 onto inside of C_1'; and $\gamma_2 := \rho_3\rho_2$ maps inside of C_2 onto the outside of C_2'. Note that one of the two fixed points of γ_1 is inside C_1', and the other one is outside C_1; one of the two fixed points of γ_2 is inside C_2', and the other one is inside C_2. The transformations γ_1 and γ_2 are conjugate to those given in §2.

We thus have four disjoint disks D_1, D_1', D_2, and D_2', whose centers are on the real axis, and two loxodromic transformations γ_1 and γ_2 taking D_1 and D_2 to the complementary disks of D_1' and D_2', respectively. Note that one of the two fixed points of γ_1 is in D_1' and the other one in D_1; one of the two fixed points of γ_2 is in D_2' and the other one in D_2.

4 The moduli space S

Two loxodromic transformations γ_1 and γ_2 generate a (classical) Schottky group if and only if there are four disjoint closed disks D_1, D_1', D_2, and D_2', such that γ_1 and γ_2 take D_1 and D_2 to the complementary disks of D_1' and D_2', respectively. Throughout the paper, disks are always assumed to be *closed*; we simply call them *disks*. The closure of the complement of a disk in $\mathbf{P}^1$ will simply be called *the complementary disk of the disk.*

Definition *Let S be the space of two loxodromic transformations γ_1 and γ_2 equipped with four disjoint disks D_1, D_1', D_2, and D_2', such that γ_1 and γ_2 take D_1 and D_2 to the complementary disks of D_1' and D_2', respectively.*

The space has a natural structure of complex 3-dimensional manifold. We are interested in its homotopic property.

Note that if a loxodromic transformation γ takes a disk D onto the complementary disk of a disk D' ($D \cap D' = \emptyset$), then γ has a fixed point in D and the other fixed point in D'.

A loxodromic transformation is determined by the two fixed points and the multiplier.

Lemma 1 *For given two disjoint disks D and D', there is a unique point $F \in D$ (resp. $F' \in D'$) such that by a(ny) linear fractional transformation taking F (resp. F') to ∞, the two circles ∂D and $\partial D'$ are transformed into cocentric circles.*

Proof. Let the two disks be given as

$$D : |z - a| \le r, \quad D' : |z - a'| \le r'.$$

By the transformation $z \to w$ defined by

$$z = \frac{1}{w} + \zeta$$

taking ζ to ∞, the circles $C = \partial D$ and $C' = \partial D'$ are transformed into circles with centers

$$c := \frac{\bar{\zeta} - \bar{a}}{r^2 - |\zeta - a|^2}, \quad c' := \frac{\bar{\zeta} - \bar{a}'}{r'^2 - |\zeta - a'|^2},$$

respectively. Equating c and c', we have the quadric equation

$$\zeta^2 + \left(-a - a' + \frac{r^2 - r'^2}{\bar{a} - \bar{a}'} \right) \zeta + aa' + \frac{ar'^2 - a'r^2}{\bar{a} - \bar{a}'} = 0.$$

Put $\zeta = (a-a')\eta + a'$. Note that $\zeta = a'$ and a correspond to $\eta = 0$ and 1, respectively. Then the equation above for ζ reduces to

$$\eta^2 + \left(-1 + \frac{r^2 - r'^2}{|a - a'|^2} \right) \eta + \frac{r'^2}{|a - a'|^2} = 0.$$

It is a high school mathematics to see that each of the two roots of this equation is in each of the two intervals

$$\left(0, \frac{r'}{|a - a'|} \right) \quad \text{and} \quad \left(1 - \frac{r}{|a - a'|}, 1 \right).$$

Since cocentric circles are mapped to cocentric circles under linear transformations, this completes the proof. []

Remark 1 *The circles ∂D and $\partial D'$ are Apollonius circles with respect to the two centers F and F'.*

Lemma 2 *For given two disjoint disks D and D' and a point f' in the interior of D', there is a 1-parameter family (parametrized by a circle) of loxodromic transformations γ which take D onto the complementary disk of D', and fix f'. Absolute value $|m|$ of the multiplier m of γ is determined by the given data.*
(1) *If $f' \ne F'$, then the other fixed point $f \in D$ of γ must be on the Apollonius circle $A = A(f')$ in D determined by the two centers of the circles ∂D and $\partial D'$, and propotion $|m|$. $\arg m \in \mathbf{R}/2\pi\mathbf{Z}$ determines f, and vice versa.*
(2) *If $f' = F'$, then the other fixed point is $F \in D$. $\arg m \in \mathbf{R}/2\pi\mathbf{Z}$ remains free.*

Proof. We can assume that $f = 0$ and $f' = \infty$, so that the transformation in question can be presented by $z \mapsto mz$. Let c and r be the center and the radius of the disk D, and c' and r' those of the complementary disk of D'. Then we have $c' = mc$ and $r' = |m|r$.
(1) If $c \neq c'$, $f (= 0)$ is on the Apollonius circle

$$A : |f - c'| = |m||f - c|$$

with centers c and c', and proportion m. It is easy to see that this circle is in D. (See Figure 2.)

Figure 2: Apollonius circle in D

(2) If $c = c' = 0$, then the Apollonius circle reduces to a point. $\square$

In the case (2), we blow up the point F to be the circle $\mathbf{R}/2\pi\mathbf{Z}$, and call this circle also the Apollonius circle A. Under this convention, two disjoint disks D and D', an interior point $f' \in D'$, and a point f on the Apollonius circle A determines uniquely a loxodromic transformation.

Thus an element of S can be determined by four disjoint disks D_1, D_1', D_2, D_2', two interior points $f_1' \in D_1', f_2' \in D_2'$, and two points f_1 on the Apollonius circle A_1 (determined by D_1, D_1', f_1') in D_1, and f_2 on the Apollonius circle A_2 (determined by D_2, D_2', f_2') in D_2. Since disks are contractible, we have

Proposition 1 *The fundamental group of S can be generated by the moves of the four disjoint disks D_1, D_1', D_2, D_2', and the moves of f_1 in A_1, and f_2 in A_2.*

In the next section we explicitly construct loops (real 1-dimensional families of classical Schottky groups) in S.

5 Loops in S

Let γ_1 and γ_2 be as in §2. The fixed points of γ_1 are $\{0, \infty\}$, and those of γ_2 are

$$f_2 = \frac{\Gamma(c)\Gamma(a-c+1)\Gamma(b-c+1)}{\Gamma(2-c)\Gamma(a)\Gamma(b)} \quad \text{and} \quad f_2' = \frac{\Gamma(c)\Gamma(1-a)\Gamma(1-b)}{\Gamma(2-c)\Gamma(c-a)\Gamma(c-b)}.$$

We change the coordinate z by multiplying $1/f_2$. Then the fixed points of γ_1 remain to be $\{0, \infty\}$, and those of γ_2 become 1 and

$$\alpha = g(a)g(b), \quad \text{where} \quad g(x) = \frac{\sin(\pi c - \pi x)}{\sin(\pi x)}.$$

Indeed we have

$$\frac{f_2'}{f_2} = \frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(c-a)\Gamma(c-b)} \cdot \frac{\Gamma(a)\Gamma(b)}{\Gamma(a-c+1)\Gamma(b-c+1)} \quad \text{and} \quad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

Definition *When the three exponent-differences are purely imaginary:*

$$\lambda = i\theta_0, \quad \mu = i\theta_1, \quad \nu = i\theta_2, \quad \theta_0, \theta_1, \theta_2 > 0,$$

the generators $\{\gamma_1, \gamma_2\}$ of the monodromy group of $E(a,b,c)$ given in §2 (take the four disks given in §3) form a simply connected real 3-dimensional submanifold of S; this submanifold is called S_0.

In this section, we construct loops in S, with base in S_0, which generate the fundamental group of S. When the exponent-differences are as above, note that the parameters can be expressed as

$$a = \frac{1}{2} - \frac{i}{2}(\theta_0 + \theta_1 + \theta_2), \quad b = \frac{1}{2} - \frac{i}{2}(\theta_0 + \theta_1 - \theta_2), \quad c = 1 - i\theta_0.$$

For clarity, we denote the four disks D_1, D_1', D_2, D_2' by

$$D_0 \ (\ni 0), \quad D_\infty \ (\ni \infty), \quad D_1 \ (\ni 1), \quad D_\alpha \ (\ni \alpha).$$

5.1 The disk D_α travels around the disk D_0

We construct a loop in S (with base in S_0), which is represented by a travel of the disk D_α around the disk D_0 with a change of argument (by 2π) of the multiplier of γ_2. We fix c and the real part of a (as 1/2), and let the real part of b move from 1/2 to 3/2; the imaginary parts of a and b are so chosen that the monodromy group $M(a,b,c)$ remains to be a Schottky group along the move. Putting $c = 1 - i\theta_0$, we have

$$g(x) = \frac{\varepsilon^{-1} - \varepsilon e^{2\pi i x}}{1 - e^{2\pi i x}} = \varepsilon + \frac{\varepsilon^{-1} - \varepsilon}{1 - e^{2\pi i x}}, \quad \text{where} \quad \varepsilon = e^{-\pi\theta_0} < 1.$$

Set

$$r = g\left(\frac{1}{2} - \frac{i}{2}(\theta_0 + 2)\right), \quad R = g\left(\frac{1}{2} - \frac{i}{2}(\theta_0 + 1)\right).$$

Then we have

$$0 < \varepsilon < r = \varepsilon\frac{e^{2\pi} + \varepsilon^{-1}}{e^{2\pi} + \varepsilon} < R = \varepsilon\frac{e^{\pi} + \varepsilon^{-1}}{e^{\pi} + \varepsilon} < 1.$$

Let D_0 be the disk with center at 0 and with radius εr, and D_∞ the complementary disk of the disk with center at 0 and with radius $\varepsilon^{-1}r$. Note that we have

$$0 < \varepsilon r < \varepsilon R < R < 1 < \varepsilon^{-1} r,$$

and that γ_1 maps D_0 onto the outside disk of D_∞.

Define real continuous functions $\phi(t)$ and $\psi(t)$ for $0 \le t \le 1$ by

$$Re^{2\pi i t} = g\left(\frac{1}{2} + \phi(t) - \frac{i}{2}(\theta_0 + \psi(t))\right).$$

Since

$$2\pi i\left(\frac{1}{2} + \phi(t) - \frac{i}{2}(\theta_0 + \psi(t))\right) = \log(Re^{2\pi i t} - \varepsilon^{-1}) - \log(Re^{2\pi i t} - \varepsilon),$$

the function $\phi(t)$ is monotone increasing with $\phi(0) = 0, \phi(1) = 1$, and $\psi(t)$ satisfies $\psi(0) = \psi(1) = 1$. Choose θ_1 so big that

- $\theta_1 > \max\{\psi(t) \mid 0 \le t \le 1\}$, and that
- for any z in the ring $\{\varepsilon R \le |z| \le R\}$, there are two disjoint disks D_1 ($\ni 1$) and D_z ($\ni z$), and a fractional linear transformation with multiplier $e^{-2\pi\theta_1}$ mapping D_1 onto the complementary disk of D_z.

Figure 3: The disk D_α travels around the disk D_0

Now set $\theta_2(t) = \theta_1 - \psi(t) > 0$ and deform the parameters a and b as

$$\begin{aligned} a(t) &= \frac{1}{2} - \frac{i}{2}(\theta_0 + \theta_1 + \theta_2(t)), \\ b(t) &= \frac{1}{2} + \phi(t) - \frac{i}{2}(\theta_0 + \theta_1 - \theta_2(t)). \end{aligned}$$

Then we have $\varepsilon < g(a(t)) < 1, |g(b(t))| = R$, and so $\alpha(t) = g(a(t))g(b(t))$ satisfies

$$\varepsilon R < |\alpha(t)| = g(a(t))|g(b(t))| < R.$$

Thus $\alpha(t)$ travels around the disk D_0 in the ring $\{\varepsilon R \le |z| \le R\}$, and there are two disjoint disks D_1 ($\ni 1$) and $D_{\alpha(t)}$ ($\ni \alpha$) in the ring $\{\varepsilon r < |z| < \varepsilon^{-1} r\}$, and a transformation $\gamma_2(t)$ with multiplier $e^{2\pi i\mu} = e^{-2\pi\theta_1}$ which maps D_1 onto the complementary disk of $D_{\alpha(t)}$.

5.2 The disk D_α travels around the disk D_1

We construct a loop in S (with base in S_0), which is represented by a travel of the disk D_α around the disk D_1 with a change of argument (by 2π) of the multiplier of γ_2. We fix c, a and the imaginary part of b, and let the real part of b move from $1/2$ to $3/2$. Take $\theta_0 > 0$ and $\theta' < 0$ satisfying

$$\frac{e^{\pi\theta'}(1 + e^{\pi\theta'})}{1 - e^{\pi\theta'}} < \varepsilon^{-2}, \quad \varepsilon = e^{-\pi\theta_0}. \tag{1}$$

Set

$$r = \varepsilon + \frac{\varepsilon^{-1} - \varepsilon}{1 + e^{\pi\theta'}}, \quad R = \varepsilon + \frac{\varepsilon^{-1} - \varepsilon}{1 - e^{\pi\theta'}}.$$

Since we have

$$\frac{1 - \varepsilon r}{r} = \frac{1 - \varepsilon^2}{r}\left(1 - \frac{1}{1 + e^{\pi\theta'}}\right) > 0$$

and

$$\varepsilon^{-1} r - \varepsilon R = (1 - \varepsilon^2)\left(1 + \frac{\varepsilon^{-2}}{1 + e^{\pi\theta'}} - \frac{1}{1 - e^{\pi\theta'}}\right) > 0,$$

where the last inequality holds thanks to (1), there is a positive number s satisfying

$$0 < s < \min\left\{r, \ \frac{1 - \varepsilon r}{r}, \ \frac{\varepsilon^{-1} r - \varepsilon R}{\varepsilon^{-1} + R}\right\},$$

that implies

$$(\varepsilon+s)r<1, \quad (\varepsilon+s)R<\varepsilon^{-1}(r-s). \tag{2}$$

Take θ_1 so big that

$$\varepsilon+\frac{\varepsilon^{-1}-\varepsilon}{1+e^{\pi(\theta_0+\theta_1)}}<\varepsilon+s. \tag{3}$$

Now set $\theta_2:=\theta_0+\theta_1-\theta'$, and

$$a=\frac{1}{2}-\frac{i}{2}(\theta_0+\theta_1+\theta_2),$$

and deform (the real part of) b as

$$b(t)=\frac{1}{2}+t-\frac{i}{2}(\theta_0+\theta_1-\theta_2)=\frac{1}{2}+t-\frac{i}{2}\theta'.$$

Note that

$$\varepsilon<g(a)=\varepsilon+\frac{\varepsilon^{-1}-\varepsilon}{1+e^{\pi(\theta_0+\theta_1+\theta_2)}}<\varepsilon+s, \quad \text{here we used (3)},$$

and that

$$g(b(t))=\varepsilon+\frac{\varepsilon^{-1}-\varepsilon}{1+e^{\pi\theta'}e^{2\pi it}} \qquad \text{satisfies} \qquad r\le|g(b(t))|\le R.$$

These inequalities together with (2) implies that $\alpha(t)=g(a)g(b(t))$ is in the ring $\{\varepsilon r\le|z|\le(\varepsilon+s)R\}$ and that

$$\alpha(0)=\alpha(1)\le(\varepsilon+s)r<1<\varepsilon R\le\alpha(1/2).$$

This shows that $\alpha(t)$ travels around 1. Let D_0 be the disk with center at 0 and with

Figure 4: The disk D_α travels around the disk D_1

radius $\varepsilon(r-s)$, D_∞ the complementary disk of the disk with center at 0 and with radius $\varepsilon^{-1}(r-s)$. Then γ_1 maps D_0 onto the complementary disk of D_∞, and by right (2), $D_0\cup D_\infty$ is disjoint from the ring

$$\{\varepsilon r\le|z|\le(\varepsilon+s)R\}\ni 1,\alpha(t).$$

Thus by taking $\theta_1>0$ sufficiently big, for any $0\le t\le 1$, there are two disjoint disks D_1 ($\ni 1$), $D_{\alpha(t)}$ ($\ni\alpha(t)$) in the complement of $D_0\cup D_\infty$ and a transformation $\gamma_2(t)$ with multiplier $e^{2\pi i\mu}=e^{-2\pi\theta_1}e^{-2\pi it}$ which maps D_1 onto the complementary disk of the $D_{\alpha(t)}$.

5.3 The multiplier of γ_2 travels around 0

We construct a loop in S (with base in S_0), which is represented by the change of argument (by 2π) of the multiplier of γ_2. We fix b and c, and the imaginary part of a, and let the real part of a move from $1/2$ to $3/2$. Set

$$\theta := \theta_0 + \theta_1 + \theta_2, \quad \varepsilon := e^{-\pi\theta_0}, \quad r := \frac{\varepsilon^{-1} - \varepsilon}{e^{\pi\theta} - 1}.$$

Choose and fix θ_0, θ_1 and θ_2 so that $\theta_1 = \theta_2$ and

$$r < \min\left\{1 - \varepsilon,\ \frac{1}{1 + \varepsilon^{-1}}\right\}.$$

Then since

$$b = \frac{1}{2} - \frac{i}{2}(\theta_0 + \theta_1 - \theta_2) = \frac{1}{2} - \frac{i}{2}\theta_0,$$

we have $e^{2\pi i b} = -\varepsilon^{-1}$, and so $g(b) = 1$.

Now we deform the parameter a as

$$a(t) = \frac{1}{2} + t - \frac{i}{2}\theta, \quad 0 \le t \le 1.$$

Since we have

$$\alpha(t) = g(a(t))g(b) = g(a(t)) = \varepsilon + \frac{\varepsilon^{-1} - \varepsilon}{1 + e^{\pi\theta}e^{2\pi i t}},$$

the point α is in the disk with center at ε and with radius r. We thus have

$$\varepsilon^2(1 + r) < \varepsilon - r \le |\alpha(t)| \le \varepsilon + r < 1 < 1 + r.$$

Let D_0 be the disk with center at 0 and with radius $\varepsilon^2(1+r)$, D_∞ the complementary

Figure 5: The fixed point α of γ_2 travels along the Apollonius circle

disk of the disk with center at 0 and with radius $1 + r$, $D_{\alpha(t)}$ ($\ni \alpha(t)$) the disk with center at ε and with radius r. Then γ_1 maps D_0 onto the complementary disk of D_∞, and $\alpha(t)$ belongs to the disk $\{|z - \varepsilon| \le r\}$ which is disjoint from $D_0 \cup D_\infty \cup \{1\}$. Thus by taking $\theta_1 = \theta_2 > 0$ sufficiently big, for any $0 \le t \le 1$, there are two disjoint disks D_1 ($\ni 1$), $D_{\alpha(t)}$ ($\ni \alpha(t)$) in the complement of $D_0 \cup D_\infty$ and a transformation $\gamma_2(t)$ with multiplier $e^{2\pi i\mu} = e^{-2\pi\theta_1}e^{-2\pi i t}$ which maps D_1 onto the complementary disk of $D_{\alpha(t)}$.

5.4 The multiplier of γ_1 travels around 0

We construct a loop in S (with base in S_0), which is represented by the change of argument (by 2π) of the multiplier of γ_1. Note that by the change of variable $x \to 1 - x$, the equation $E(a, b, c)$ changes into $E(a, b, a + b + 1 - c)$. So we have only to literally follow §5.3 exchanging c and $a + b + 1 - c$, and θ_0 and θ_1. We thus fix b and the imaginary parts of a and c, and let the real part of c move from 1 to 2, and let the real part of a move from 1/2 to 3/2 keeping $a + b + 1 - c$ constant.

6 Miscellanea

For a Schottky group Γ of genus 2, the quotient of the domain of discontinuity modulo Γ is a curve of genus 2. A curve of genus 2 is a double cover of $\mathbf{P}^1$ branching at six points, which are uniquely determined modulo automorphisms of $\mathbf{P}^1$ by the curve. Thus a Schottky group determines a point of the configuration space $X\{6\}$ of six-point sets on $\mathbf{P}^1$. (The fundamental group of $X\{6\}$ is the Braid group with five strings.) When all the exponent-differences of the hypergeometric equation are real, its monodromy group is a Schottky group, which determines six points on a line. Thus they determines a point of the configuration space $X(6)$ of colored six point on $\mathbf{P}^1$. (The fundamental group of $X(6)$ is the colored Braid group with five strings.) The space $X(6)$ is well-studied.

Remark 2 *When all the exponent-differences are real, there is a Schottky automorphic function defined by an absolutely convergent infinite product, which induces a holomorphic map of the genus 2 curve onto* $\mathbf{P}^1$ ([IY]). *This infinite product remains to be convergent for groups represented by loops above, if we take the multipliers of* γ_1 *and* γ_2 *sufficiently small. This is because there is a circle separating two disks among the four* ([BBEIM, Chapter 5]).

Problems: The four loops constructed in §5 induces those in $X\{6\}$. Do they generate the fundamental group of $X\{6\}$? Same problem for $X(6)$.

Acknowledgement: The authors are grateful to Professors Susumu Hirose, Ken'ichi Ohshika, Hiroki Sato and Hiro-o Yamamoto for their encouragement.

References

[BBEIM] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its and V. B. Matveev, *Algebro-geometric Approach to Nonlinear Integrable Equations,* Springer Series in Nonlinear Dynamics (Springer-Verlag, 1994).

[IY] T. Ichikawa and M. Yoshida, On Schottky groups arising from the hypergeometric equation with imaginary exponents, Proc AMS 132(2003), 447–454.

[SY] T. Sasaki and M. Yoshida, A geometric study of the hypergeometric fucntion with imaginary exponents, Experimental Math. **10**(2000), 321 –330.

[Sch] F. Schottky, Über eine specielle Function, welche bei einer bestimmten linearen Transformation ihres Arguments univerändert bleibt, J. Reine Angew. Math. 101 (1887) 227-272.